NOTE ON CLASSICAL HAMILTONIAN THEORY*

(ZAMECHANIIA O KLASSICHESKOI GAMIL'TONOVOI THEORII)

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The requirement that deviations of theoretical values from the natural values of observed functions be small imposes a series of conditions on the forces in a strict theoretical sense.

In rough engineering theories the requirement is that the undisturbed motion must be asymptotically stable in order that the equations of the disturbed motion represent the instrument or system to the first approximation. Then all the exciting forces of order of smallness above the first will not be able to make the undisturbed motion unstable [1]. But this elementary consideration interests me little now.

Actual exciting forces, if they do make the undisturbed motion unstable, are detected by inadmissible deviations between theoretical and natural values of observed functions, with respect to which the undisturbed motion becomes unstable; and consequently such forces must be introduced into the theory if the latter is to satisfy the requirement of small deviations of theory from experiment. In a rigorous theory, real exciting forces must not cause a well established stable equilibrium or an undisturbed motion of a mechanical system to become unstable.

In mechanics we possess the brilliant theory of holonomic mechanical systems acted upon by forces derived from a force function.

This theory has been well tested. The equations of motion of such mechanical systems in the canonical variables q_s, p_s are written in the Hamiltonian form

$$\frac{dq_s}{dt} = \frac{\partial H}{\partial p_s}, \qquad \frac{dp_s}{dt} = -\frac{\partial H}{\partial q_s} \qquad (s = 1, \ldots, n)$$

where $H(t, q_1, \ldots, q_n, p_1, \ldots, p_n)$ is the Hamiltonian function.

* This work was discovered in the writings of N.G. Chetaev and dated.

The first approximation to the equations of the disturbed motions have the form of equations in Poincaré's variations

$$\frac{d\xi_s}{dt} = \sum \left(\frac{\partial^2 H}{\partial p_s \partial q_j} \xi_j + \frac{\partial^2 H}{\partial p_s \partial p_j} \eta_j \right), \qquad \frac{d\eta_s}{dt} = - \sum \left(\frac{\partial^2 H}{\partial q_s \partial q_j} \xi_j + \frac{\partial^2 H}{\partial q_s \partial p_j} \eta_j \right)$$

where ξ_s, η_s are corresponding deviations of the coordinates q_s and the momenta p_s.

The stability of the predominant or undisturbed motion may be assured only when the characteristic numbers of all solutions of the variational equations are zero [2].

If the undisturbed motion is stable, then the equations in Poincaré variations appear to lead to a system of equations with constant coefficients [3] and consequently [4] arbitrarily small exciting forces may make stable motions unstable. But why, in virtue of all this, has the Hamiltonian theory so well justified itself?

Let us consider the simplest and most frequently treated case, in which the undisturbed motion represents the equilibrium of a mechanical holonomic system under the action of forces given by a force function.

What is here the force barrier which protects the system against large deviations under the action of arbitrarily small exciting forces? The answer to this was found by Lord Kelvin.

Normal coordinates exist close to the equilibrium position; to the first approximation the equations of the disturbed motion close to the stable position have the following form:

$$x_i'' = -a_i x_i \quad \text{for positive } a_i$$

Besides the general exciting forces which are assumed to be of a higher order of smallness than the first, there are also assumed to be present linear small dissipative forces with components

$$X_i = -\frac{\partial f}{\partial x_s'}$$

where the dissipation function $f = \Sigma c_{\alpha\beta} x_\alpha' x_\beta'$ represents a positive definite quadratic form with constant coefficients, in the velocities x_j'.

If the equilibrium is stable under potential forces, then it becomes asymptotically stable under additional dissipative forces with complete dissipation, as well as for exciting forces of a higher order of smallness than the first [5].

All this leads us to conclude that small dissipative forces with complete dissipation, always present in nature, are the force barriers which make negligible any effects due to nonlinear exciting forces.

Similar considerations apply to the stable undisturbed motions of a Hamiltonian system, since in this case the equations reduce to Poincaré variations also.

Problems of the exciting forces may be extremely diverse and may be different from those just considered.

In the paper "On Stable Trajectories in Dynamics" the problem of exciting potential forces was considered, with one determining condition that the stability of the undisturbed motion of a mechanical system did not depend on the potential of the assumed exciting forces but on the force function of the given forces and on the physically significant constant *vis viva* of the system.

The problem as postulated has been solved within the framework of classical mechanics.

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